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Conference Paper *in* AIP Conference Proceedings · November 2017

DOI: 10.1063/1.5012154

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On the Cyclic Decomposition of Complete Multigraph Into Near Hamiltonian Cycles

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Abstract. Let v and λ be positive integer, λK_v denote a complete multigraph. A decomposition of a graph G is a set of subgraphs of G whose edge sets partition the edge set of G. In this article, difference set method is used to introduce a new design that is decomposed a complete multigraph into near Hamiltonian cycles. In course of developing this design, a combination between near-4-factor and a cyclic $(v-1)$-cycle system of $4K_v$, when $v = 4n+2, n > 2$, will be constructed.
Keywords: complete multigraph, (cyclic) m-cycle system, near-k-factor, difference set.

1. Introduction

In our paper, all graphs consider finite and undirected. A complete graph of order v denotes by K_v. An m-cycle, written $C_m = (c_0, \dots, c_{m-1})$, consists of m distinct vertices $\{c_0, c_1, \dots, c_{m-1}\}$ and m edges $\{c_i c_{i+1}\}, 0 \leq i \leq m-2$ and $c_0 c_{m-1}$. An m-path, written $[c_0, \dots, c_{m-1}]$, consists of m distinct vertices $\{c_0, c_1, \dots, c_{m-1}\}$ and $m-1$ edges $\{c_i c_{i+1}\}, 0 \leq i \leq m-2$.

An m-cycle system of a graph G, called a decomposition of G into m-cycles or (G, C_m)-design, is a pair (V, C) where V is the vertex set of G and C is a collection of edge-disjoint of m-cycles whose edges partitions the edge set of G. If $G = K_v$ then such an m-cycle system is called m-cycle system of order v. A m-cycle system is Hamiltonian if $m = |V|$. It is a cyclic if $V = Z_v$ and we have $C_m + 1 = (c_0 + 1, c_1 + 1, \dots, c_{m-1} + 1) \in C$ whenever $C_m = (c_0, c_1, \dots, c_{m-1}) \in C$ and is said to be simple when its cycles are all distinct.

A complete multigraph of order v, denoted by λK_v, is obtained by replacing each edge of K_v with λ edges. A m-cycle system of λK_v is a collection of m-cycles whose edges partitions of $E(\lambda K_v)$ the edge multi-set of λK_v. The necessary and sufficient conditions for the existence of m-cycle system of λK_v have been established by Bryant et al. in [2].For the important case of $\lambda = 1$, the existence question for m-cycle system of K_v has been completely settled by Alspach and Gavlas [1] in the case m odd and by Šajna [6] in the case m even. While, for the existence question for cyclic m-cycle system of order v has been solved for $m = 3$ denoted by $\mathrm{CSS}(v, \lambda)$ [5] and for a cyclic Hamiltonian cycle system of order v was proved when v is an odd integer but $v \neq 15$ and $v \neq p^\alpha$ with p a prime and $\alpha > 1$ [3].

A k-factor of a graph G is a spanning subgraph whose vertices have a degree k. While, a near-k-factor is a subgraph in which all vertices has a degree k with exception of one vertex (isolated vertex) which has a degree zero.

In this paper we propose a new cycle system that is called cyclic near Hamiltonian cycle system of $4K_v$, denoted $CNHC(4K_v, C_{v-1})$. This is obtained by combination a cyclic $(v-1)$-cycle system of $4K_v$, when $v \equiv 2(mod\ 4)$, and near-4-factor. In addition, $CNHC(4K_v, C_{v-1})$ is an $(v \times 2)$ array that satisfies the following conditions:

- The cycles in row r and column i form a near-4-facto with focus r.
- The cycle associated with the rows contain no repetition.

2. Preliminaries

Throughout the paper all graphs and cycles considered have vertices in Z_v and $Z_v^* = Z_v - \{0\}$. Let $G = \lambda K_v$, when v is even, the difference D of edge $\{ab\} \in E(\lambda K_v)$ is defined as $D(a,b) = min\{|a-b|, v-|a-b|,\}$, arithmetic $(mod\ v)$. So that, the difference of any edge in $E(\lambda K_v)$ is less than or equal to $v/2$, $(1 \le d \le v/2)$. Give $C_m = (c_0, \dots, c_{m-1})$ a m-cycle, the list of difference from C_m is a multiset $D(C_m) = \{min\{|a_i - a_{i-1}|, v - |a_i - a_{i-1}|\}|i = 1, 2, \dots, m\}$ where $a_0 = a_m$. Let $\mathcal{F} = \{B_1, B_2, \dots, B_r\}$ be an m-cycles of λK_v is called $(\lambda K_v, C_m)$-difference system of λK_v if the multiset $D(\mathcal{F}) = \bigcup_{i=1}^{r} D(B_i)$ covers each element of $Z_{\frac{v}{2}}^*$ exactly λ times and the middle difference $\left(\frac{v}{2}\right)$ appear $\left\{\frac{\lambda}{2}\right\}$ times.

The orbit of cycle C_m, denoted by $orb(C_m)$, is the set of all distinct m-cycles in the collection $\{C_m + i | i \in Z_v\}$. The length of $orb(C_m)$ is its cardinality, i.e., $orb(C_m) = k$ where k is the minimum positive integer such that $C_m + k = C_m$. A cycle orbit of length v on λK_v is said to be full and otherwise short. A set of m-cycles that generated the cyclic m-cycle system of λK_v by repeated addition of 1 modular v is called base cycles.

For presenting a cyclic m-cycle system of λK_v, it sufficient to give a set of base cycle of it. As a particular consequence of the theory developed in [4] we have:

Proposition 2.1 A set $\mathcal{F} = \{B_1, B_2, \dots, B_r\}$ of m-cycle is a base cycles of cyclic m-cycle system of λK_v if and only if $\mathcal{F}$ is $(\lambda K_v, C_m)$-difference system of λK_v.

3. Near Hamiltonian Cyclic System

Definition 3.1 For $v = 4n + 2, n > 2$, a full cyclic near Hamiltonian cycle system of the $4K_v$ graph denoted by $CNHC(4K_v, C_{v-1})$, is a Cyclic$(v-1)$-cycle system of $4K_v$ graph, that satisfies the following conditions:

- The cycle in row r form a near-4-factor with focus r.
- The cycle associated with the rows contain no repetitions.

Surely, for presenting a full cyclic near Hamiltonian cycle system of the $4K_v$, $CNHC(4K_v, C_{v-1})$, it is sufficient to provide a set of base cycles that satisfies a near-4-factor. We give here example to explain the above definition.

Example 3.1 let $G = 4K_{14}$ and $\mathcal{F} = \{C_{13}, C_{13}^*\}$ is a set of 13-cycles of G such that
$C_{13} = (1, 13, 2, 12, 3, 11, 4, 5, 10, 6, 9, 7, 8), C_{13}^* = (13, 8, 12, 9, 11, 10, 4, 3, 5, 2, 6, 1, 7)$
Firstly, it is easy to observe that each non zero element in Z_{14} occurs exactly twice in the -cycles of $\mathcal{F}$. So that, every vertex has a degree 4 except a zero element (isolated vertex) has a degree zero. Then, it is satisfies the near-4-factor. Secondly, the list of difference set for the cycles in $\mathcal{F}$ is listed in TABLE 3.1**.**

TABLE 3.1

13-cycle	**Difference set**
$(1, 13, 2, 12, 3, 11, 4, 5, 10, 6, 9, 7, 8)$	$\{2, 3, 4, 5, 6, 7, 1, 5, 4, 3, 2, 1, 7\}$
$(13, 8, 12, 9, 11, 10, 4, 3, 5, 2, 6, 1, 7)$	$\{5, 4, 3, 2, 1, 6, 1, 2, 3, 4, 5, 6, 6\}$

It can be seen from the TABLE 3.1 $D(\mathcal{F}) = D(C_{13}) \cup D(C_{13}^*)$ the list of difference set of $\mathcal{F}$ covers each integer in Z_7^* exactly four times and the middle difference 7 twice. Therefore, the set $\mathcal{F} = \{C_{13}, C_{13}^*\}$ is a $(4K_{14}, C_{13})$-difference system of $4K_{14}$. Then, by Proposition 2.1 the cycles of the set $\mathcal{F}$ are the base cycles of $CNHC(4K_{14}, C_{13})$.

Then, $CNHC(4K_{14}, C_{13})$ is an (14×2) array design and the base cycles $\mathcal{F} = \{C_{13}, C_{13}^*\}$ in the first row generate all cycles in (14×2) array by repeated 1 modular 14 as shown in the TABLE 3.2

TABLE 3.2

Focus	$\mathbf{CNHC(4K_{14}, C_{13})}$	
i = 0	$(1, 13, 2, 12, 3, 11, 4, 5, 10, 6, 9, 7, 8)$	$(13, 8, 12, 9, 11, 10, 4, 3, 5, 2, 6, 1, 7)$
i = 1	$(2, 0, 3, 13, 4, 12, 5, 6, 11, 7, 10, 8, 9)$	$(0, 9, 13, 10, 12, 11, 5, 4, 6, 3, 7, 2, 8)$
i = 2	$(3, 1, 4, 0, 5,13, 6, 7, 12, 8, 11, 9, 10)$	$(1, 10, 0, 11, 13, 12, 6, 5, 7, 4, 8, 3, 9)$
⋮	⋮	⋮
i = 13	$(0, 12, 1, 11, 2, 10, 3, 4, 9, 5, 8, 6, 7)$	$(12, 7, 11,8, 10, 9, 3, 2, 4, 1, 5, 0, 6)$

Throughout the paper, a near Hamiltonian cycle of order $(v - 1)$ will be represented as connected paths, we mean that $C_{v-1} = \left(c_{(1)}, P_{(1,2)}^{2n}, P_{(3,4)}^{2n}\right)$ where, $P_{(1,2)}^{2n}$ and $P_{(3,4)}^{2n}$ are $(2n)$-paths such that:

$P_{(1,2)}^{2n} = \left[c_{(1,1)}, c_{(2,1)}, c_{(1,2)}, c_{(2,2)}, \dots, c_{(1,n)}, c_{(2,n)}\right]$.

$P_{(3,4)}^{2n} = \left[c_{(3,1)}, c_{(4,1)}, c_{(3,2)}, c_{(4,2)}, \dots, c_{(3,n)}, c_{(4,n)}\right]$.

Let the vertex sets of $P_{(1,2)}^{2n}$ and $P_{(3,4)}^{2n}$ are $\left\{\bigcup_{i=1}^{n} c_{(1,i)}, \bigcup_{i=1}^{n} c_{(2,i)}\right\}$, $\left\{\bigcup_{i=1}^{n} c_{(3,i)}, \bigcup_{i=1}^{n} c_{(4,i)}\right\}$, respectively. And the list of difference sets of $P_{(1,2)}^{2n}$ and $P_{(3,4)}^{2n}$ will be calculated as follows:

$D\left(P_{(1,2)}^{2n}\right) = D_1\left(P_{(1,2)}^{2n}\right) \cup D_2\left(P_{(1,2)}^{2n}\right)$, $D\left(P_{(3,4)}^{2n}\right) = D_1\left(P_{(3,4)}^{2n}\right) \cup D_2\left(P_{(3,4)}^{2n}\right)$ such that

$D_1\left(P_{(1,2)}^{2n}\right) = \left\{min\left\{\left|c_{(1,i)} - c_{(2,i)}\right|, v - \left|c_{(1,i)} - c_{(2,i)}\right|\right\} \middle| 1 \le i \le n\right\}$.

$D_2\left(P_{(1,2)}^{2n}\right) = \left\{min\left\{\left|c_{(1,i+1)} - c_{(2,i)}\right|, v - \left|c_{(1,i+1)} - c_{(2,i)}\right|\right\} \middle| 1 \le i \le n-1\right\}$.

$D_1\left(P_{(3,4)}^{2n}\right) = \left\{min\left\{\left|c_{(3,i)} - c_{(4,i)}\right|, v - \left|c_{(3,i)} - c_{(4,i)}\right|\right\} \middle| 1 \le i \le n\right\}$.

$D_2\left(P_{(3,4)}^{2n}\right) = \left\{min\left\{\left|c_{(3,i+1)} - c_{(4,i)}\right|, v - \left|c_{(3,i+1)} - c_{(4,i)}\right|\right\} \middle| 1 \le i \le n-1\right\}$.

And we define $\mathrm{D}\left(c_{(1)}, P_{(1,2)}^{2n}\right) = D(c_{(1)}, c_{(1,1)})$, $D\left(P_{(3,4)}^{2n}, c_{(1)}\right) = D(c_{(4,n)}, c_{(1)},)$ and $D\left(P_{(1,2)}^{2n}, P_{(3,4)}^{2n}\right) = D(c_{(2,n)}, c_{(3,1)})$. So, the list of difference of C_{v-1} is represented as the follows:
$D(C_{v-1}) = D\left(P_{(1,2)}^{2n}\right) \cup D\left(P_{(3,4)}^{2n}\right) \cup D\left(c_{(1)}, P_{(1,2)}^{2n}\right) \cup D\left(P_{(3,4)}^{2n}, c_{(1)}\right) \cup D\left(P_{(1,2)}^{2n}, P_{(3,4)}^{2n}\right)$.
Now we are able to provide our main result.

Theorem 3.1 There are exists a full cyclic near Hamiltonian cycle system of $4K_v$, $CNHC(4K_v, C_{v-1})$, when $v = 4n + 2, n > 2$.

Proof. Suppose $\mathcal{F} = \{C_{4n+1}, C_{4n+1}^*\}$ is the set of base cycles of $4K_{4n+2}$ where

$C_{4n+1} = \left(1, P_{(1,2)}^{2n}, P_{(3,4)}^{2n}\right)$, $C_{4n+1}^* = \left(2n+1, P_{(1,2)}^{(2n)^*}, P_{(3,4)}^{(2n)^*}\right)$ such that:

- $P_{(1,2)}^{2n} = [4n + 1, 2, 4n, 3, \dots, 3n + 2, n + 1]$.
- $P_{(3,4)}^{2n} = [n + 2, 3n + 1, n + 3, 3n, \dots, 2n + 1, 2n + 2]$.
- $P_{(1,2)}^{(2n)^*} = [4n + 1, 2n + 2, 4n, 2n + 3, \dots, 3n + 2, 3n + 1]$.
- $P_{(3,4)}^{(2n)^*} = [n + 1, n, n + 2, n - 1, \dots, 2n, 1]$.

We will divide the proof into two parts as follows:

Part. 1 In this part will be proved that $\mathcal{F}$ satisfies a near-4-factor. We will calculate the vertex set of C_{4n+1} and C_{4n+1}^* such that:

$V(C_{4n+1}) = V\left(P_{(1,2)}^{2n}\right) \cup V\left(P_{(3,4)}^{2n}\right) \cup \{1\}$

$V(C_{4n+1}^*) = V\left(P_{(1,2)}^{(2n)^*}\right) \cup V\left(P_{(3,4)}^{(2n)^*}\right) \cup \{2n + 1\}$

- $\bigcup_{i=1}^{n} c_{(1,i)} = \{4n+2-i, \quad 1 \le i \le n\} = \{4n+1, 4n, \dots, 3n+2\}$.
- $\bigcup_{i=1}^{n} c_{(2,i)} = \{i+1 \qquad , \quad 1 \le i \le n\} = \{2, 3, \dots, n+1\}$.
- $\bigcup_{i=1}^{n} c_{(3,i)} = \{n+1+i, \quad 1 \le i \le n\} = \{n+2, n+3, \dots, 2n+1\}$.
- $\bigcup_{i=1}^{n} c_{(4,i)} = \{3n+2-i \ , \quad 1 \le i \le n\} = \{3n+1, 3n, \dots, 2n+2\}$.

Then $V(C_{4n+1})$ cover each nonzero element of Z_{4n+2} exactly once.

- $\bigcup_{i=1}^{n} c^*_{(1,i)} = \{4n+2-i \ , \quad 1 \le i \le n\} = \{4n+1, 4n, \dots, 3n+2\}$.
- $\bigcup_{i=1}^{n} c^*_{(2,i)} = \{2n+1+i \ , \quad 1 \le i \le n\} = \{2n+2, 2n+3, \dots, 3n+1\}$.
- $\bigcup_{i=1}^{n} c^*_{(3,i)} = \{n+i \qquad , \quad 1 \le i \le n\} = \{n+1, n+2, \dots, 2n\}$.
- $\bigcup_{i=1}^{n} c^*_{(4,i)} = \{n+1-i\} = \{n, n-1, \dots, 1\}$.

It can be observed from the above equations that $V(C_{4n+1}) = V(C^*_{4n+1})$. Then, the multiset $V(C_{4n+1}) \cup V(C^*_{4n+1})$ covers each nonzero elements of Z_{4n+2} exactly twice. Consequently, $\mathcal{F} = \{C_{4n+1}, C^*_{4n+1}\}$ satisfies a near -4 - factor (with isolated zero).

Part. 2 In this part we will prove $\mathcal{F} = \{C_{4n+1}, C^*_{4n+1}\}$ is the base cycles of cyclic $(v-1)$-cycle system of $4K_v$. So, we will calculate the difference set of each of them as follows:

$$D(C_{4n+1}) = D\left(P^{2n}_{(1,2)}\right) \cup D\left(P^{2n}_{(3,4)}\right) \cup D\left(c_{(1)}, P^{2n}_{(1,2)}\right) \cup D\left(P^{2n}_{(1,2)}, P^{2n}_{(3,4)}\right) \cup D\left(P^{2n}_{(3,4)}, c_{(1)}\right).$$

- $D_1\left(P^{2n}_{(1,2)}\right) = \bigcup_{i=1}^{n}(2i+1) = \{3, 5, \dots, 2n+1\}$.
- $D_2\left(P^{2n}_{(1,2)}\right) = \bigcup_{i=1}^{n-1}(2i+2) = \{4, 6, \dots, 2n\}$.
- $D_1\left(P^{2n}_{(3,4)}\right) = \bigcup_{i=1}^{n}(2n+1-2i) = \{2n-1, 2n-3, \dots, 1\}$.
- $D_2\left(P^{2n}_{(3,4)}\right) = \bigcup_{i=1}^{n-1}(2n-2i) = \{2n-2, 2n-4, \dots, 2\}$.
- $D\left(c_{(1)}, P^{2n}_{(1,2)}\right) = D\left(c_{(1)}, c_{(1,1)}\right) = D(1, 4n+1) = \{2\}$,
- $D\left(P^{2n}_{(1,2)}, P^{2n}_{(3,4)}\right) = D\left(c_{(2,n)}, c_{(3,1)}\right) = D(n+1, n+2) = \{1\}$.
- $D\left(P^{2n}_{(3,4)}, c_{(1)}\right) = D\left(c_{(4,n)}, c_{(1)}\right) = D(2n+2, 1) = \{2n+1\}$.

We note that the list of difference of C_{v-1}, $D(C_{v-1})$, covers each integers of Z^*_{2n+2} twice except $\{2n\}$ once. Now we will calculate $D(C^*_{4n+1})$ such as:

$$D(C^*_{4n+1}) = D\left(P^{(2n)^*}_{(1,2)}\right) \cup D\left(P^{(2n)^*}_{(3,4)}\right) \cup D\left(c^*_{(1)}, P^{(2n)^*}_{(1,2)}\right) \cup D\left(P^{(2n)^*}_{(1,2)}, P^{(2n)^*}_{(3,4)}\right) \cup \quad D\left(P^{(2n)^*}_{(3,4)}, c^*_{(1)}\right).$$

- $D_1\left(P^{(2n)^*}_{(1,2)}\right) = \bigcup_{i=1}^{n}(2n+1-2i) = \{2n-1, 2n-3, \dots, 1\}$.
- $D_2\left(P^{(2n)^*}_{(1,2)}\right) = \bigcup_{i=1}^{n-1}(2n-2i) = \{2n-2, 2n-4, \dots, 2\}$.
- $D_1\left(P^{(2n)^*}_{(3,4)}\right) = \bigcup_{i=1}^{n}(2i-1) = \{1, 3, \dots, 2n-1\}$.
- $D_2\left(P^{(2n)^*}_{(3,4)}\right) = \bigcup_{i=1}^{n-1}(2i) = \{2, 4, \dots, 2n-2\}$.
- $D\left(c^*_{(1)}, P^{(2n)^*}_{(1,2)}\right) = D\left(c^*_{(1)}, c^*_{(1,1)}\right) = D(2n+1, 4n+1) = \{2n\}$.
- $D\left(P^{(2n)^*}_{(1,2)}, P^{(2n)^*}_{(3,4)}\right) = D\left(c^*_{(2,n),} c^*_{(3,1)}\right) = D(3n+1, n+1) = \{2n\}$.
- $D\left(P^{(2n)^*}_{(3,4)}, c^*_{(1)}\right) = D\left(c^*_{(4,n)}, c^*_{(1)}\right) = D(1, 2n+1) = \{2n\}$.

As clearly shown, in the previous equations, every non zero integers in Z^*_{2n+1} appears twice except $\{2n\}$ appears three times in $D(C^*_{4n+1})$.

One can easily observe that $D(\mathcal{F}) = D(C_{4n+1}) \cup D(C^*_{4n+1})$ the list of differences of $\mathcal{F} = \{C_{4n+1}, C^*_{4n+1}\}$ covers each non zero integers in Z_{2n+2} four times except the middle difference $\{2n+1\}$ twice. Thus, $\mathcal{F} = \{C_{4n+1}, C^*_{4n+1}\}$ is a $(4K_{4n+1}, C_{4n+1})$-difference system of $4K_{4n+2}$. By Proposition 2.1, the cycles of the set $\mathcal{F}$are the base cycles of the full near Hamiltonian cycle system of $4K_v$when $v = 4n + 2,\ n > 2$ □

ACKNOWLEDGMENTS

I am thankful to University Utara Malaysia for holding like this conference which provides a platform for the students and researchers to share their ideas, experiences and research results in various areas of mathematics and statistic.

References

1. B. Alspach and H. Gavlas, Cycle decompositions of Kn and Kn− I. Journal of Combinatorial Theory, Series B, 81(1), 77-99 (2001)..
2. D. Bryant, D. Horsley, B. Maenhaut and B. R. Smith, Cycle decompositions of complete multigraphs, Journal of Combinatorial Designs, 19(1), 42-69 (2011).
3. M. Buratti and A. Del Fra, Cyclic Hamiltonian cycle systems of the complete graph, Discrete mathematics, 279(1), 107-119 (2004).
4. M. Buratti, A description of any regular or 1-rotational design by difference methods, Booklet of the abstracts of Combinatorics, 35-52 (2000).
5. M. J. Colbourn and C. J. Colbourn, Cyclic block designs with block size 3, European Journal of Combinatorics, 2(1), 21-26 (1981).
6. M. Šajna, Cycle decompositions III: complete graphs and fixed length cycles, Journal of Combinatorial Designs, 10(1), 27-78 (2002).